JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **33**, No. 3, August 2020
http://dx.doi.org/10.14403/jcms.2020.33.3.327

ON STRONGLY RIGHT π-DUO RINGS

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Abstract. This article continues the study of right π-duo rings, concentrating on the situation of nonzero powers. For this purpose we introduce the concept of *strongly right π-duo* and examine the structure of strongly right π-duo in relation to various ring properties that play important roles in ring theory. It is proved for a strongly right π-duo ring R that if the upper (lower) nilradical of R is zero then R is reduced. Various kinds of examples are examined in relation to the questions raised in the procedure.

Throughout this article every ring is associative with identity unless otherwise specified. Given a ring R (possibly without identity), $J(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ denote the Jacobson radical, the prime radical, the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in R, respectively. It is well-known that $N^*(R) \subseteq J(R)$ and $N_*(R) \subseteq N^*(R) \subseteq N(R)$. We use $R[x]$ to denote the polynomial ring with an indeterminate x over R. Denote the n by n full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Use e_{ij} for the matrix unit with (i,j)-entry 1 and elsewhere 0. Denote $\{(a_{ij}) \in U_n(R) \mid \text{ the diagonal entries of } (a_{ij}) \text{ are all equal}\}$ by $D_n(R)$. $r_R(-)$ (resp., $l_R(-)$) is used to denote a right (resp., left) annihilator in R. $\prod$ denotes the direct product of rings. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n).

Following Feller [2], a ring (possibly without identity) is called *right duo* if every right ideal is two-sided. Left duo rings are defined similarly. A ring is called duo if it is both left and right duo. Following Kim et al. [6], a ring R (possibly without identity) is *right π-duo* provided that for any $a \in R$, there exists a positive integer n such that $Ra^n \subseteq aR$. Left π-duo rings are defined similarly. A ring is called *π-duo* if it is both left and right π-duo. A ring (possibly without identity) is usually

Received October 30, 2019; Accepted June 09, 2020.
2010 Mathematics Subject Classification: 16D25, 16N40, 16U80.
Key words and phrases: strongly right π-duo ring, right duo ring, right π-duo ring, weakly right duo ring, nilradical, nilpotent element.
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called *Abelian* if every idempotent is central. Left or right π-duo rings are Abelian by [6, Proposition 1.9(4)]. Yao [11] called a ring R (possibly without identity) *weakly right duo* if for each a in R there exists a positive integer n such that a^nR is a two-sided ideal of R. Weakly left duo rings are defined similarly. A ring is called *weakly duo* if it is both weakly left and weakly right duo. It is obvious that right (resp., left) duo rings are weakly right (resp., left) duo and weakly right (resp., left) duo rings are right (resp., left) π-duo (hence Abelian). It is well-known that every implication is irreversible.

1. Strongly right π-duo rings

In this section, we introduce the concept of a strongly right π-duo ring as a special case of right π-duo ring, and study the basic structure of strongly right π-duo rings.

A ring R shall be called *strongly right π-duo* provided that for any $0 \neq a \in R$, there exists a positive integer k such that

$$a^k \neq 0 \text{ and } Ra^k \subseteq aR.$$

The strongly left π-duo rings are defined analogously, and strongly π-duo means both strongly right and strongly left π-duo. It is clear that right duo rings are strongly right π-duo and strongly right π-duo rings are right π-duo. Each of these implications is irreversible as follows. We will use this fact freely throughout this note. The following shows that right π-duo rings need not be strongly right π-duo.

Example 1.1. Let S be any ring and $R = D_n(S)$ for $n \geq 3$. Assume that R is strongly right π-duo. Let $A = e_{2n}$. Since R is strongly right π-duo and $A^2 = 0$, we must get

$$RA = Se_{1n} + Se_{2n} \subseteq AR = Se_{2n},$$

a contradiction. Thus R is not strongly right π-duo. R being not strongly left π-duo is also shown by a similar method.

Let S be a division ring. Then R is weakly duo (hence π-duo) since every element is either invertible or nilpotent.

Example 1.1 also shows that weakly right duo rings need not be strongly right π-duo, comparing this with the fact that weakly right duo rings are right π-duo.

The strongly π-duo property is not left-right symmetric by the following.

Example 1.2. There is a strongly right π-duo ring that is not (strongly) left π-duo. We refer to the ring in [5, Example 1]. Let $F(x)$ be the quotient filed of the polynomial ring $F[x]$ with an indeterminate x over a field F. Define a ring endomorphism $\sigma : F(x) \to F(x)$ by $\sigma(f(x)/g(x)) = f(x^2)/g(x^2)$. We consider the skew power series ring $R = F(x)[[t;\sigma]]$ with the elements $\sum_{i=0}^{\infty} t^i k_i$ for $k_i \in F(x)$. The multiplication is only subject to $kt = t\sigma(k)$ for $k \in F(x)$. We first note that each coefficient of the elements in Rt^n is of the form $f(t^{2n})/g(t^{2n})$. Then R is not left π-duo by the argument in [6, Example 1.3], so that R is not strongly left π-duo. In fact, $t^m x \notin Rt$ for any $m \geq 1$.

Next we will show that R is right duo (hence strongly right π-duo), by applying the method in [5, Example 1]. Let $0 \neq f(t) \in R$. Then $f(t) = \sum_{i=k}^{\infty} t^i a_i$ with $a_k \neq 0$ for some $k \geq 0$. Let $g(t) = \sum_{j=0}^{\infty} t^j b_j$ with $b_j = a_{j+k}$ for all j. Then $g(t)$ is invertible in R and $f(t) = t^k g(t)$. Hence we have $h(t)f(t) = h(t)t^k g(t) = t^k h'(t)g(t) = t^k g(t)g(t)^{-1}h'(t)g(t) = f(t)g(t)^{-1}h'(t)g(t) \in f(t)R$ for every $h(t) \in R$, where $t^k h'(x) = h(t)t^k$. Therefore R is right duo.

The ring R in Example 1.1 is weakly right duo but not strongly right π-duo. We next consider a strongly right π-duo ring but not weakly right duo, entailing that strongly right π-duo and weakly right duo are independent of each other.

Example 1.3. We use the ring and argument in [7] which is based on the construction of [5, Example 4]. Let $F = \mathbb{Z}_2$ and $S = F[t]$ be the polynomial ring with an indeterminate t over F. Consider a ring homomorphism $\sigma : S \to S$ by $\sigma(f(t)) = f(t^2)$, and the skew polynomial ring $T_0 = S[x;\sigma]$ over S by σ, in which every element is of the form $\sum_{i=0}^{m} x^i a_i$ with $a_i \in S$, only subject to $sx = x\sigma(s)$ for all $s \in S$. Let $T_1 = T_0/x^2T_0$ and identify each element of T_0 with its image in T_1 for simplicity. Note that $\sigma(f(t)) = f(t)^2$ for all $f(t) \in S$, and every element of T_1 is of the form $s_0 + xs_1$ with $s_i \in S$. Set next

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in S \text{ and } b \in xT_1 \right\},$$

a subring of $D_2(T_1)$. Then R is not weakly right duo by the argument in [7].

We will show that R is strongly right π-duo. Let $D = \begin{pmatrix} a & xe \\ 0 & a \end{pmatrix} \in R$ and $g = xe$ with $e \in T_1$. If $a \neq 0$ then we have $0 \neq RD^3 \subseteq DR$ by the

argument in [7]. Consider next the case of $a = 0$, i.e., $D = \begin{pmatrix} 0 & xe \\ 0 & 0 \end{pmatrix}$.
For any $E = \begin{pmatrix} f(t) & b \\ 0 & f(t) \end{pmatrix} \in R$, we get

$$ED = \begin{pmatrix} f(t) & b \\ 0 & f(t) \end{pmatrix} \begin{pmatrix} 0 & xe \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x\sigma(f(t))e \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & xe \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f(t)^2 & 0 \\ 0 & f(t)^2 \end{pmatrix} \in DR$$

when $f(t) \neq 0$. If $f(t) = 0$ then $ED = 0 \in DR$. Therefore R is strongly right π-duo.

The following diagram shows all implications among the concepts above.

$$\begin{array}{ccc} & \boxed{\text{strongly right } \pi\text{-duo rings}} & \\ \nearrow & & \searrow \\ \bullet \text{ right duo rings} & & \text{right } \pi\text{-duo rings} \\ \searrow & & \nearrow \\ & \text{weakly right duo rings} & \end{array}$$

The following contains basic properties of strongly right π-duo rings that play important roles throughout. A ring R (possibly without identity) is usually called *reduced* if it has no nonzero nilpotent elements. Reduced rings are easily shown to be Abelian.

Proposition 1.4. *For a strongly right π-duo ring R, we have the following.*

(1) *Every finite direct product of strongly right π-duo rings is also strongly right π-duo.*

(2) *Let $e \in R$ be a central idempotent of a ring R. Then R is strongly right π-duo if and only if eR and $(1-e)R$ are strongly right π-duo rings.*

(3) *$N(R) \subseteq J(R)$. If $J(R)$ is nil then $R/J(R)$ is a reduced ring.*

Proof. (1) Let R_i, $i \in \{1, \ldots, n\}$, be strongly right π-duo rings, and $R = \prod_{i=1}^n R_i$. Take $0 \neq a = (a_1, \ldots, a_n) \in R$. Then there exist $i_1, \ldots, i_l$ such that $a_{i_j} \neq 0$ for all $j = 1, \ldots, l$. Since every R_i is strongly right π-duo, there exist $k_{i_1}, \ldots, k_{i_l}$ such that $0 \neq R_{i_j} a_{i_j}^{k_{i_j}} \subseteq a_{i_j} R_{i_j}$. Now let k be maximal in $\{k_{i_1}, \ldots, k_{i_l}\}$. Then $R_i a_i^k \subseteq a_i R_i$ for all i and $R_i a_i^k$ is nonzero for some i. This yields that $0 \neq Ra^k \subseteq aR$, showing that $R = \prod_{i=1}^n R_i$ is strongly right π-duo.

(2) Suppose that R is strongly right π-duo. For $0 \neq er \in eR$, there exists $k \geq 1$ such that $0 \neq R(er)^k \subseteq erR$. Since e is central, we also

have

$$eR(er)^k = R(er)^k \subseteq erR = (er)eR,$$

entailing that eR is strongly right π-duo. Similarly, $(1-e)R$ is also strongly right π-duo. The converse comes from (1).

(3) Recall that for a (strongly) right π-duo ring R, aR and Ra are both nil for all $a \in N(R)$ by [6, Theorem 2.1(1)]. Thus, since every nil one-sided ideal is contained in the Jacobson radical, $N(R)$ is contained in $J(R)$. Next suppose that $J(R)$ is nil. Then we get $J(R) = N(R)$ since $N(R) \subseteq J(R)$. Thus $R/J(R)$ is reduced. □ □

Following [3], a ring R is said to be *von Neumann regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. It is easily checked that von Neumann regular rings are semiprimitive.

PROPOSITION 1.5. *Let R be a von Neumann regular ring. The following are equivalent:*
(1) *R is right duo.*
(2) *R is strongly right π-duo.*
(3) *R is right π-duo.*
(4) *R is Abelian.*

Proof. (1)⇒(2)⇒(3) come from definitions, (3)⇒(4) is shown by [6, Proposition1.9 (4)], and (4)⇒(1) follows from [3, Theorem 3.2]. □ □

Recall that a ring R is called *local* if $R/J(R)$ is a division ring, and R is *semilocal* if $R/J(R)$ is semisimple Artinian. A ring R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are Abelian and semilocal.

PROPOSITION 1.6. *A ring R is strongly right π-duo and semiperfect if and only if R is a finite direct product of local strongly right π-duo rings.*

Proof. Suppose that R is strongly right π-duo and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [8, Proposition 3.7.2], say $R = \prod_{i=1}^n e_iR$ such that each e_iRe_i is a local ring. By [6, Proposition1.9 (4)], R is Abelian and so $e_iR = e_iRe_i$ for each i. But each e_iR is also strongly right π-duo by Proposition 1.4(2).

Conversely assume that R is a finite direct product of local strongly right π-duo rings. Then R is semiperfect since local rings are semiperfect by [8, Corollary 3.7.1], and moreover R is strongly right π-duo by Proposition 1.4(1). □ □

Due to Marks [9], a ring R is called *NI* if $N^*(R) = N(R)$. Note that R is NI if and only if $N(R)$ forms an ideal of R if and only if $R/N^*(R)$ is reduced. Duo rings are NI, but not conversely. A ring R is said to be *of bounded index* (*of nilpotency*) if there exists a positive integer n such that $a^n = 0$ for all $a \in N(R)$.

PROPOSITION 1.7. *Assume that R is strongly right π-duo ring. Then we have the following results.*

(1) $N^*(R) = 0$ *if and only if* $N(R) = 0$.

(2) $N_*(R) = 0$ *if and only if* $N(R) = 0$.

(3) *If R is a ring of bounded index* 2, *then R is NI.*

Proof. We freely use the strongly right π-duo property of R.

(1) Assume on the contrary that $N^*(R) = 0$ and $N(R) \neq 0$. Let $0 \neq a \in N(R)$. Then aR is nil by [6, Theorem 2.1(1)], and $0 \neq Ra^k \subseteq aR$ for some $k \geq 1$. These are combined with the result that

$$0 \neq Ra^kR \subseteq aR \subseteq N(R),$$

entailing $a^k \in N^*(R)$. Thus $N^*(R) \neq 0$, a contradiction. So $N(R) = 0$. The converse is obvious.

(2) It suffices to show the necessity. Let $N_*(R) = 0$, i.e. R is a semiprime ring. Assume on the contrary that $a^2 = 0$ for some $0 \neq a \in R$. Then $Ra \subseteq aR$, so that $aRa \subseteq a^2R = 0$. Since R is semiprime, we get $a = 0$, a contradiction. Thus $N(R) = 0$.

(3) Let R be a ring of bounded index 2. If $N(R) = 0$ then we are done. Suppose $N(R) \neq 0$ and let $0 \neq a \in N(R)$. Then $0 \neq RaR$ must be contained in aR because $a^2 = 0$. Since aR is nil by [6, Theorem 2.1(1)], we have $RaR \subseteq N^*(R)$. This entails $a \in N^*(R)$ and so $N(R) = N^*(R)$.
□ □

Reduced rings are Abelian. So, considering Proposition 1.7, one may ask whether Abelian NI rings are strongly right π-duo. But the answer is negative by the following.

EXAMPLE 1.8. Let A be an Abelian NI ring (e.g., reduced rings) and $n \geq 3$. Set $R = D_n(A)$. Then R is Abelian by [4, Lemma 2], and NI since $N^*(R) = \{(a_{ij}) \in R \mid a_{ii} = 0\}$ and $R/N^*(R) \cong A$. But R is neither strongly right nor strongly left π-duo by Example 1.1.

2. Examples of strongly right π-duo rings

In this section, we investigate several kinds of ring extensions that can be strongly right π-duo. We first observe an equivalent relation between strongly right π-duo and right duo, through a kind of matrix ring. There exist many domains which are neither right nor left duo. For example, a free algebra with two or more noncommuting inderminates over a field is neither right nor left duo. Recall that $D_n(S)$ is not strongly right π-duo for all $n \geq 3$ over any ring S by Example 1.1.

PROPOSITION 2.1. *Let R be a reduced ring. Then R is right duo if and only if $D_2(R)$ is strongly right π-duo.*

Proof. Suppose that R is right duo. Let $0 \neq A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in D_2(R)$. If $a = 0$ then $A = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ and $A^2 = 0$. For any $X = \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \in D_2(R)$, we have

$$XA = \begin{pmatrix} 0 & rb \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & br' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r' & s \\ 0 & r' \end{pmatrix},$$

where $rb = br'$ for some $r' \in R$ by the right duoness of R. Thus $XA \in AD_2(R)$, concluding $D_2(R)A \subseteq AD_2(R)$.

Next suppose $a \neq 0$ and $k \geq 2$. Then $a^k \neq 0$ because R is reduced, so that $A^k \neq 0$. Since R is right duo, $0 \neq Ra^k \subseteq a^kR \subseteq aR$. Let $r \in R$. Then $ra^k = a^k r_k$ for some $r_k \in R$.

For any $s \in R$, we get

$$\begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^k = \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \begin{pmatrix} a^k & b_1 \\ 0 & a^k \end{pmatrix} = \begin{pmatrix} ra^k & rb_1 + sa^k \\ 0 & ra^k \end{pmatrix} = \begin{pmatrix} a^k r_k & a\alpha \\ 0 & a^k r_k \end{pmatrix}$$

where $rb_1 + sa^k \in RaR$ and $rb_1 + sa^k = a\alpha$ for some $\alpha \in R$ because $RaR \subseteq aR$.

Moreover $ba^{k-1}r_k = a\beta$ for some $\beta \in R$ since $Ra^{k-1}R \subseteq RaR \subseteq aR$. Thus we now have

$$\begin{aligned} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} a^{k-1}r_k & \alpha - \beta \\ 0 & a^{k-1}r_k \end{pmatrix} &= \begin{pmatrix} a^k r_k & a\alpha - a\beta + ba^{k-1}r_k \\ 0 & a^k r_k \end{pmatrix} \\ &= \begin{pmatrix} a^k r_k & a\alpha - a\beta + a\beta \\ 0 & a^k r_k \end{pmatrix} = \begin{pmatrix} a^k r_k & a\alpha \\ 0 & a^k r_k \end{pmatrix} \\ &= \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^k, \end{aligned}$$

concluding $0 \neq D_2(R)A^k \subseteq AD_2(R)$. Therefore $D_2(R)$ is strongly right π-duo.

Conversely suppose that $D_2(R)$ is strongly right π-duo. Let $0 \neq a \in R$ and consider $A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \in D_2(R)$. Then since $A^2 = 0$ and $D_2(R)$ is strongly right π-duo, for any $\begin{pmatrix} b & c \\ 0 & b \end{pmatrix} \in D_2(R)$, there exists $\begin{pmatrix} b' & c' \\ 0 & b' \end{pmatrix} \in D_2(R)$ such that

$$\begin{pmatrix} 0 & ba \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & c \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b' & c' \\ 0 & b' \end{pmatrix} = \begin{pmatrix} 0 & ab' \\ 0 & 0 \end{pmatrix},$$

entailing $ba = ab'$. But b is arbitrary in R and $Ra \subseteq aR$ follows. Thus R is right duo. □ □

Considering Proposition 2.1, one may ask whether $D_2(R)$ is strongly right π-duo over a strongly right π-duo ring R. But the answer is negative as follows.

Example 2.2. Let R be the right duo ring R in Example 1.2. Then R is strongly right π-duo by Proposition 2.1. Let

$$A = \begin{pmatrix} 0 & \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \\ 0 & 0 \end{pmatrix}, \ B = \begin{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \end{pmatrix} \in D_2(R).$$

Then $BA = \begin{pmatrix} 0 & \begin{pmatrix} 0 & tx \\ 0 & 0 \end{pmatrix} \\ 0 & 0 \end{pmatrix}$ and $A^2 = 0$. Assume that there exists $C = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \in D_2(R)$ and $k \geq 1$ such that $A^k \neq 0$ and $BA^k = AC$, where $\alpha, \beta \in R$. But $A^2 = 0$, forcing $k = 1$. Let $\alpha = \begin{pmatrix} f(t) & g(t) \\ 0 & f(t) \end{pmatrix}$ with $f(t) = \sum_{i=0}^{\infty} t^i k_i$. Then, from $BA = AC$, we get $tx = xf(t) = \sum_{i=0}^{\infty} t^i (x^{2i} k_i)$, a contradiction. Thus $D_2(R)$ is not strongly right π-duo.

The strongly right π-duo ring property does not go up to polynomial rings and the class of strongly right π-duo rings is not closed under subrings by help of [6, Proposition 1.9(1)].

Example 2.3. The strongly right π-duo ring property does not go up to polynomial rings. Let R be a noncommutative division ring. Then R

is clearly strongly π-duo. But $R[x]$ is not right π-duo by [6, Proposition 1.9(1)] and [5, Proposition 8]. Hence $R[x]$ is not strongly right π-duo.

The class of strongly right π-duo rings is not closed under subrings. Note that $R[x]$ is Noetherian domain and so the classical quotient ring of $R[x]$ is a division ring by [10, Corollary 2.1.14 and Theorem 2.1.15]. Thus the classical quotient ring of $R[x]$ is strongly π-duo, but the subring $R[x]$ is not strongly right π-duo.

But we have information when polynomial rings are strongly right π-duo.

Proposition 2.4. *For a ring R, if $R[x]$ is strongly right π-duo then so is R.*

Proof. We apply the proof of [6, Proposition 2.4]. Let $R[x]$ be strongly right π-duo and $0 \neq a \in R$. Then $a^k \neq 0$ and $R[x]a^k \subseteq aR[x]$ for some $k \geq 1$. So for $r \in R$, $(r + x)a^k = a(b + cx)$ for some $b + cx \in R[x]$. This yields $ra^k = ab$, so that $Ra^k \subseteq aR$. □ □

Let A be an algebra with an identity over a commutative ring S. Following Dorroh [1], the *Dorroh extension* of A by S is the Abelian group $A \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$.

Proposition 2.5. *Let A be an algebra with an identity over a commutative ring S. If the Dorroh extension of A by S is strongly right π-duo then A is strongly right π-duo.*

Proof. Note that $s \in S$ is identified with $s1 \in A$, and so $A = \{r + s \mid (r, s) \in D\}$. Let D be the Dorroh extension of A by S. Suppose that D is strongly right π-duo and let $0 \neq a \in A$. Then there exists $k \geq 1$ such that $(a, 0)^k \neq 0$ and $D(a, 0)^k \subseteq (a, 0)D$. Then $a^k \neq 0$ and thus for any $r \in A$,

$$(ra^k, 0) = (r, 0)(a, 0)^k = (a, 0)(r_1, s_1) = (ar_1 + as_1, 0)$$

for some $(r_1, s_1) \in D$. This yields $ra^k = a(r_1 + s_1) \in aA$, noting $s_1 = s_11 \in A$. Therefore A is strongly right π-duo. □ □

The converse of Proposition 2.5 need not hold. Let $R = D_n(\mathbb{Z})$ for $n \geq 3$. Then R is neither strongly right nor strongly left π-duo by Example 1.1. But R is an algebra with identity over $\mathbb{Z}$ and R is isomorphic to the Dorroh extension of R by $\mathbb{Z}$, noting $n = nI_n$ for all $n \in \mathbb{Z}$, where I_n is the identity matrix of R.

Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* elements can be defined. An element is *regular* if it is both left and right regular (and hence not a zero divisor).

PROPOSITION 2.6. *Let M be a multiplicatively closed subset of a ring R consisting of central regular elements. If R is strongly right π-duo, then $M^{-1}R$ is strongly right π-duo.*

Proof. Suppose that R is strongly right π-duo and let $u^{-1}a \in M^{-1}R$ with $a \neq 0$. Since R is strongly right π-duo, there exists $k \geq 1$ such that $a^k \neq 0$ and $Ra^k \subseteq aR$. Then $(u^{-1}a)^k \neq 0$. Let $\alpha \in M^{-1}R(u^{-1}a)^k$. Then $\alpha = (v^{-1}b)(u^{-1}a)^k = v^{-1}(u^{-1})^k ba^k$ for some $v^{-1}b \in M^{-1}R$. Since $ba^k \in Ra^k \subseteq aR$, $ba^k = ac$ for some $c \in R$. Then

$$\alpha = v^{-1}(u^{-1})^{k-1}u^{-1}ba^k = w^{-1}u^{-1}ac = (u^{-1}a)w^{-1}c \in (u^{-1}a)M^{-1}R,$$

where $v^{-1}(u^{-1})^{k-1} = w^{-1}$. Thus $M^{-1}R$ is strongly right π-duo. □ □

Recall the ring of *Laurent polynomials* in x, written by $R[x; x^{-1}]$. Letting $M = \{1, x, x^2, \ldots\}$, M is clearly a multiplicatively closed subset of central regular elements in $R[x]$ such that $R[x; x^{-1}] = M^{-1}R[x]$. So Proposition 2.6 yields the following.

COROLLARY 2.7. *Let R be a ring. If $R[x]$ is strongly right π-duo then $R[x; x^{-1}]$ is strongly right π-duo.*

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